L16 Conditions for UMP with two-sided H_a

1. Three types of tests covered before Exam 2

   (1) Simple H_0 and simple H_a
   Suppose sample X has pdf/pmf $f(X;\theta)$ and $\Lambda = \frac{f(X;\theta_1)}{f(X:\theta_0)}$.

   $$H_0: \theta = \theta_0 \text{ versus } H_a: \theta = \theta_1$$
   $$\phi(X) = \begin{cases} 1 & \Lambda > k \\ r & \Lambda = k \\ 0 & \Lambda < k \end{cases} \text{ with } \alpha = E_{\theta_0}[\phi(X)]$$
   is α-level MP test

   (2) Simple H_0 and one-sided H_a
   Suppose likelihood has monotone ratio in $T(X)$, i.e., $\Lambda = \frac{f(X;\theta_2)}{f(X;\theta_1)}$ is an increasing function of $T(X)$ for all $\theta_1 < \theta_2$.

   $$H_0: \theta = \theta_0 \text{ vs } H_a: \theta < \theta_0$$
   $$\phi(T) = \begin{cases} 1 & T < k \\ r & T = k \\ 0 & T > k \end{cases}$$
   with $\alpha = E_{\theta_0}[\phi(T)]$
   is α-level UMP

   $$H_0: \theta = \theta_0 \text{ vs } H_a: \theta > \theta_0$$
   $$\phi(T) = \begin{cases} 1 & T > k \\ r & T = k \\ 0 & T < k \end{cases}$$
   with $\alpha = E_{\theta_0}[\phi(T)]$
   is α-level UMP

   (3) One-sided H_0 and one-sided H_a
   Based on the study on $\beta_\phi(\theta)$ for tests in (2), we find that the tests in (2) are also α-level UMP tests for $H_0: \theta \geq \theta_0$ versus $H_a: \theta < \theta_0$ and $H_0: \theta \leq \theta_0$ versus $H_a: \theta > \theta_0$.

2. Problems in Exam 2

   (1) For $H_0: \theta = \theta_0$ vs $H_a: \theta = \theta_1$

x	1	2	3	4	5
$f(x;\theta_0)$	0.2	0.2	0.2	0.2	0.2
$f(x;\theta_1)$	0.1	0.4	0.1	0.1	0.3
$\Lambda = f_1(x)/f_0(x)$	0.5	2	0.5	0.5	1.5

   $$\Phi(X) = \begin{cases} 1 & \Lambda > k \\ r & \Lambda = k \\ 0 & \Lambda < k \end{cases}$$

   $0.05 = P_{\theta_0}(\Lambda > K) + r\,P_{\theta_0}(\Lambda = K)$.

   $r = \frac{0.05 - P_{\theta_0}(\Lambda > k)}{P_{\theta_0}(\Lambda = k)} = \frac{0.05 - P_{\theta_0}(\Lambda > 2)}{P_{\theta_0}(\Lambda = 2)} = \frac{0.05-0}{0.2} = 0.25.$

   Thus $\phi(X) = \begin{cases} 0.25 & X = 2 \\ 0 & X = 1, 3, 4, 5 \end{cases}$ is MP test among all tests with level 0.05.

   (2) In 3 of Exam 2, $X_1, ..., X_n$ is a random sample from $N(0, \sigma^2)$.

   (i) Find T such that the likelihood function has monotone likelihood ratio in T.
   $\sigma_1^2 < \sigma_2^2 \Longrightarrow \frac{f(X;\sigma_2^2)}{f(X;\sigma_1^2)} = \cdots =$ is an increasing function of $T(X) = \sum_i X_i^2$

   (ii) Find the distribution of T
   $X_1, ..., X_n$ iid $N(0, \sigma^2) \Longrightarrow \frac{\sum X_i^2}{\sigma^2} \sim \chi^2(n) \Longrightarrow T(X) = \sum X_i^2 \sim \sigma^2\chi^2(n)$.

(iii) For $H_0: \sigma^2 \le 4$ vs $H_a: \sigma^2 > 4$ find α-level UMP test.

$\phi(X) = \begin{cases} 1 & T = \sum_{i=1}^n X_i^2 > k \\ 0 & T = \sum_{i=1}^n X_i^2 < k \end{cases}$

$$\alpha = E_{\sigma^2=4}[\phi(X)] = P_{\sigma^2=4}(T > k) = P(4\chi^2(n) > k) = P\left(\chi^2(n) > \frac{k}{4}\right),$$

So $\frac{k}{4} = \chi^2_\alpha(n) \Longrightarrow k = 4\chi^2_\alpha(n)$.

Thus $\phi(X) = \begin{cases} 1 & \sum_{i=1}^n X_i^2 > 4\chi^2_\alpha(n) \\ 0 & \sum_{i=1}^n X_i^2 < 4\chi^2_\alpha(n) \end{cases}$ is α-level UMP test.

3. A type of test covered after Exam 2

(1) Two-sided H_a tests

$f(X;\theta) = \exp[p(\theta) + q(X) + \theta T(X)]$. So sufficient T has $f(t;\theta) = a(\theta)b(t)e^{\theta t}$.

> $H_0: \theta = \theta_0$ versus $H_a: \theta \ne \theta_0$
> $\phi(T) = \begin{cases} 1 & T < c_1 \text{ or } T > c_2 \\ r & T = c_1 \text{ or } T = c_2 \\ 0 & c_1 < T < c_2 \end{cases}$
> with $\int_t \phi(t) f(t;\theta_0)\,dt = \alpha$ and $\int_t \phi(t)[f(t;\theta_0)]'_\theta\,dt = 0$
> is UMP in α-level unbiased test class

(2) Two conditions for $\phi(T)$

$\int_t \phi(t) f(t;\theta_0)\,dt = \alpha \Longleftrightarrow E_{\theta_0}[\phi(T)] = \alpha$.

$$\begin{aligned} 0 &= \int_t \phi(t)[f(t;\theta_0)]'_\theta\,dt = \int_t \phi(t)[a(\theta_0)b(t)e^{\theta_0 t}]'_\theta\,dt \\ &= \int_t \phi(t)\left[\frac{a'(\theta_0)}{a(\theta_0)} f(t;\theta_0) + t f(t;\theta_0)\right]dt = \frac{a'(\theta_0)}{a(\theta_0)}\alpha + E_{\theta_0}[T\phi(T)] \end{aligned}$$

So $\int_t \phi(t)[f(t;\theta_0)]'_\theta\,dt \Longleftrightarrow \int_t \phi(t) t f(t;\theta_0)\,dt = -\frac{a'(\theta_0)}{a(\theta_0)}\alpha \Longleftrightarrow E_{\theta_0}[T\phi(T)] = -\frac{a'(\theta_0)}{a(\theta_0)}\alpha$.

This is one of the problems in HW07

Hence the two conditions for $\phi(T)$ are $\begin{cases} E_{\theta_0}[\phi(T)] = \alpha \\ E_{\theta_0}[T\phi(T)] = -\frac{a'(\theta_0)}{a(\theta_0)}\alpha \end{cases}$

Ex: Let $\psi(T) \equiv \alpha$. Then $\beta_\psi(\theta_0) = \alpha$ and $[\beta_\psi(\theta_0)]'_\theta = 0$, i.e.,

$$\int_t \psi(t) f(t;\theta_0)\,dt = \alpha \text{ and } \int_t \psi(t) t f(t;\theta_0)\,dt = -\frac{a'(\theta_0)}{a(\theta_0)}\alpha$$

where the second equation

$$\begin{aligned} \int_t \psi(t) t f(t;\theta_0)\,dt = -\frac{a'(\theta_0)}{a(\theta_0)}\alpha &\Longleftrightarrow \int_t t f(t;\theta_0)\,dt\,\alpha = -\frac{a'(\theta_0)}{a(\theta_0)}\alpha \\ &\Longleftrightarrow \int_t t f(t;\theta_0)\,dt = -\frac{a'(\theta_0)}{a(\theta_0)} \\ &\Longleftrightarrow E_{\theta_0}(T) = -\frac{a'(\theta_0)}{a(\theta_0)} \end{aligned}$$

Thus the two conditions for UMP $\phi(T)$ can also be written as

$$\begin{cases} E_{\theta_0}[\phi(T)] = \alpha \\ E_{\theta_0}[T\phi(T)] = \alpha E_{\theta_0}(T) \end{cases}$$

L17: Examples of UMP with two-sided H_a

1. UMP test of α-level unbiased tests

   (1) Modified $\phi(T)$
   Suppose random sample has joint pdf/pmf $f(x;\,\theta) = \exp[p(\theta) + q(x) + \theta T(x)]$. Then

   > For $H_0:\,\theta=\theta_0$ versus $H_a:\,\theta\neq\theta_0$
   >
   > $$\phi(T) = \begin{cases} 1 & T<c_1 \text{ or } T>c_2 \\ r_i & T=c_i,\ i=1,\,2 \\ 0 & c_1<T<c_2 \end{cases}$$
   >
   > with $E_{\theta_0}[\phi(T)]=\alpha$ and $E_{\theta_0}[T\phi(T)]=\alpha E_{\theta_0}(T)$
   > is UMP α-level unbiased test.

   (2) Generalization I
   Suppose $f(x;\,\theta) = \exp[p(\theta) + q(x) + \eta(\theta)T(x)]$ where $\eta(\theta)$ is a 1-1 function.

   (i) By reparameterization $\eta=\eta(\theta)$ with $\eta_0=\eta(\theta_0)$,

   > For $H_0:\,\eta=\eta_0$ versus $H_a:\,\eta\neq\eta_0$
   >
   > $$\phi(T) = \begin{cases} 1 & T<c_1 \text{ or } T>c_2 \\ r_i & T=c_i,\ i=1,\,2 \\ 0 & c_1<T<c_2 \end{cases}$$
   >
   > with $E_{\eta_0}[\phi(T)]=\alpha$ and $E_{\eta_0}[T\phi(T)]=\alpha E_{\eta_0}(T)$
   > is UMP α-level unbiased test.

   (ii) Note that $\eta=\eta_0 \Longleftrightarrow \theta=\theta_0$ and $E_{\eta_0}(\cdot)=E_{\theta_0}(\cdot)$.

   > For $H_0:\,\theta=\theta_0$ versus $H_a:\,\theta\neq\theta_0$
   >
   > $$\phi(T) = \begin{cases} 1 & T<c_1 \text{ or } T>c_2 \\ r_i & T=c_i,\ i=1,\,2 \\ 0 & c_1<T<c_2 \end{cases}$$
   >
   > with $E_{\theta_0}[\phi(T)]=\alpha$ and $E_{\theta_0}[T\phi(T)]=\alpha E_{\theta_0}(T)$
   > is UMP α-level unbiased test.

2. An example with discrete distribution
   (Modified from 6.8 (a) on p175) $X_1,...,X_n$ is a random sample from Bernoulli(θ). For

   $$H_0:\,\theta=0.2 \text{ versus } H_a:\,\theta\neq 0.2$$

   find UMP test in α-level unbiased test class with $\alpha=0.1$. Assume $n=10$.

   (1) A general form
   The joint pmf of the sample is

   $$\begin{aligned} f(x;\,\theta) &= \prod_i \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} = \left(\frac{\theta}{1-\theta}\right)^{\sum x_i}(1-\theta)^n \\ &= \exp\left[n\ln(1-\theta) + \left(\ln\frac{\theta}{1-\theta}\right)\sum x_i\right] \end{aligned}$$

   where $\eta=\ln\frac{\theta}{1-\theta}$ is a 1-1 function of θ and $T=\sum X_i \sim B(n,\,\theta)$. Thus

For $H_0: \theta = 0.2$ versus $H_a: \theta \neq 0.2$

$$\phi(T) = \begin{cases} 1 & T < c_1 \text{ or } T > c_2 \\ r_i & T = c_i,\ i = 1,\ 2 \\ 0 & c_1 < T < c_2 \end{cases}$$

with $E_{\theta=0.2}[\phi(T)] = 0.1$ and $E_{\theta=0.2}[T\phi(T)] = 0.1 \times E_{\theta=0.2}(T)$ is UMP test in α-level unbiased test class with $\alpha = 0.2$.

(2) Find c_1, c_2, r_1 and r_2

With $T \sim B(10,\ 0.2)$, by the first condition on $\phi(T)$,

$$0.1 = P(T < c_1) + P(T > c_2) + r_1 P(T = c_1) + r_2 P(T = c_2).$$

So, $P(T < c_1) \leq 0.1 \Longrightarrow c_1 \leq 0 \Longrightarrow c_1 = 0$. The first condition becomes

$$0.1 = P(T > c_2) + 0.10737 r_1 + r_2 P(T = c_2).$$

So, $P(T > c_2) \leq 0.1 \Longrightarrow c_2 \geq 4$.

With $c_2 = 4$, the first condition becomes $0.10737 r_1 + 0.08808 r_2 = 0.06721$ and the second condition, $E_{0.2}[T\phi(T)] = 0.1 \times E_{0.2}(T)$, becomes

$$4 r_2 P(T = 4) + \sum_{k=5}^{10} k P(T = k) = 0.1 \times 2, \text{ i.e.,}$$

$$\begin{aligned} 0.2 &= 4\gamma_2 \times 0.0881 + 5 \times 0.0264 + 6 \times 0.0055 + 7 \times 0.0008 + 8 \times 0.0001 \\ &= 0.3524\gamma_2 + 0.1714. \end{aligned}$$

Solve the equations. We have $\gamma_2 = 0.0812$ and $\gamma_1 = 0.5591$.

Thus $c_1 = 0$, $c_2 = 4$, $r_1 = 0.5591$ and $r_2 = 0.0812$.

3. An example with continuous distribution

$X_1, ..., X_n$ is a random sample from $N(\mu,\ \sigma_0^2)$. For

$$H_0: \mu = \mu_0 \text{ versus } H_a: \mu \neq \mu_0$$

find UMP test in α-level unbiased test class

(1) General form

$f(x;\ \mu) = \cdots = \exp[p(\mu) + q(x) + \eta(\mu) Z(x)]$ where $\eta(\mu) = \frac{\mu\sqrt{n}}{\sigma_0}$ is a 1-1 function of μ, and $Z(x) = \frac{\overline{X} - \mu_0}{\sigma_0/\sqrt{n}} \sim N(0,\ 1^2)$ under $H_0: \mu = \mu_0$. So

$$\phi(Z) = \begin{cases} 0 & c_1 < Z < c_2 \\ 1 & Z < c_1 \text{ or } Z > c_2 \end{cases} \quad \text{with } E_{\mu_0}[\phi(Z)] = \alpha \text{ and } E_{\mu_0}[Z\phi(Z)] = 0$$

(2) By Condition 2 on $\phi(Z)$, $E_{\mu_0}[Z\phi(Z)] = \alpha E_{\mu_0}(Z) = \alpha \times 0 = 0$.

So $c_1 = -c$ and $t_2 = c$.

(3) By Condition 1 on $\phi(Z)$, $c = Z_{\alpha/2}$.

(4) Result

$H_0: \mu = \mu_0$ versus $H_a: \mu \neq \mu_0$

Test Statistic: $Z = \frac{\overline{X} - \mu_0}{\sigma_0/\sqrt{n}}$

Reject H_0 if $Z < -Z_{\alpha/2}$ or $Z > Z_{\alpha/2}$